Romesh K. Kaul and S. Kalyana Rama *
The Institute of Mathematical Sciences, Chennai 600 113, India.

Recent suggestion, that the emission of a quantum of energy corresponding to the asymptotic value of quasinormal modes of a Schwarzschild black hole should be associated with the loss of spin one punctures from the black hole horizon, fixes the Immirzi parameter to a definite value. We show that saturating the horizon with spin one punctures reproduces the earlier formula for the black hole entropy, including the $ln(area)$ correction with definite coefficient $-3/2$ for large area.

One of the major achievements of the canonical quantum gravity (or quantum geometry) framework has been confirming the Bekenstein-Hawking area law for the entropy of large black holes [1,2], originally obtained from semi-classical considerations. It has also been argued that an area law follows on the basis of some symmetry principles of the semiclassical theory without any reference to detailed properties of actual quantum states of a black hole [3].

Another aspect of quantum black holes is the discrete spectrum of the horizon area, first conjectured in the pioneering work of Bekenstein [4]. The canonical quantum gravity framework does indeed predict discrete spectra for quantum area and volume operators. The Hilbert space of canonical gravity is described by spin networks, where SU(2) spins ($j = 0, 1/2, 1, \cdots$) reside on the edges of the network graph. For a surface intersected by edges of this spin network, each puncture carrying spin j contributes an amount of area equal to $A(j) = 8\pi l_p^2 \beta \sqrt{j(j+1)}$ [5] where l_p is the Planck length and β is the Immirzi parameter [6].

A detailed understanding of quantum degrees of freedom of a $(3+1)-$dimensional black hole is presented in terms of an SU(2) Chern-Simons theory on the horizon, with coupling k proportional to the horizon area [2,7–9]. Boundary states of a Schwarzschild black hole are charcterised by an $SU(2)_k$ Wess-Zumino conformal field theory on this two sphere (spatial slice of the horizon) of area A_H. The dimensionality of the boundary Hilbert space then can be calculated exactly by counting the number of conformal blocks of this two dimensional conformal field theory with a number of punctures produced by the edges of the spin network ending on this two sphere. This number for a set of punctures at p locations $\{1, 2, \cdots, p\}$ with spins $j_1, j_2, \cdots, j_p$ residing on them is given by the formula [7]

$$\mathcal{N} = \frac{2}{k+2} \sum_{r=0}^{k/2} \frac{\prod_{i=1}^{p} sin\left(\frac{(2j_i+1)(2r+1)\pi}{k+2}\right)}{\left[sin\left(\frac{(2r+1)\pi}{k+2}\right)\right]^{p-2}} \,. \tag{1}$$

For a given horizon area A_H, the largest number of states correspond to saturating the horizon by placing the lowest spin, namely spin $\frac{1}{2}$, on each puncture. In reference [8], the formula (1) was thus evaluated, with $j_i = j_{min} = \frac{1}{2}$ for all i, in the limit of large k and large p to be

$$\mathcal{N} \simeq \frac{2^p}{p^{\frac{3}{2}}} \left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right) \,. \tag{2}$$

Since the associated area for each spin $\frac{1}{2}$ puncture is $8\pi l_p^2 \beta \sqrt{\frac{3}{4}}$, the number of punctures for a fixed horizon area A_H is given by $p = \frac{\beta_0}{\beta}\left(\frac{A_H}{4l_p^2}\right)$ where $\beta_0 = (\pi\sqrt{3})^{-1}$. This when substituted in (2) yields for the entropy of the black hole [8] $S = ln\mathcal{N} = \frac{A_H}{4l_p^2} - \frac{3}{2}\, ln\left(\frac{A_H}{4l_p^2}\right) + \cdots,$ where we have used the identification for Immirzi parameter $\beta = \beta_0 ln2 = \frac{ln2}{\pi\sqrt{3}}$ to match the coefficient of the first term to the Bekenstein-Hawking area law. The $ln(area)$ correction term with definite coefficient $-\frac{3}{2}$ has also been obtained by Carlip by exploiting the nature of corrections to the Cardy formula for the density of states in a two dimensional conformal field theory [10]. This correction appears for a variety of black holes independent of dimensions. In particular, the same correction obtains for BTZ black holes in $(2+1)-$dimensional theories [10–12]. As emphasised by Carlip, this suggests a possible universal character of the $ln(area)$ correction. Same $ln(area)$ correction with coefficient $-3/2$ has also been obtained by Gour [13] in the algebraic approach of [14].

Recently, a case has been made by Dreyer [15] that the minimum spin value to be placed on each of the puncture should be $j_{min} = 1$ instead of the value $\frac{1}{2}$. This conclusion is based on a crucial observation made by Hod [16] that the real part (ringing frequencies) ω_{QNM}^R of the (complex) quasinormal mode frequencies $\omega_{QNM} \equiv \omega_{QNM}^R + i\omega_{QNM}^I$ for

*email: kaul, krama@imsc.res.in

a Schwarzschild black hole of mass M, earlier obtained numerically by Nollert [17] and later confirmed by Andersson [18], has an asymptotic value

$$\omega^R_{QNM} = \frac{ln3}{8\pi M} \tag{3}$$

for large damping, namely for large values of the imaginary part ω^I_{QNM} of ω_{QNM}. More recently, Hod's observation has been confirmed analytically [19]. Bohr's correspondence principle then suggests that ω^R_{QNM} should appear as a transition frequency in the quantum theory. This in turn implies, using the area-mass relation $A_H = 16\pi M^2$, that the area of a black hole can change by an amount $\Delta A_H = 32\pi M\Delta M$ by emitting or absorbing a quantum of energy $\Delta M = \hbar\omega^R_{QNM}$ so that $\Delta A_H = 4l_p^2 ln3$ [16]. On the other hand, the area associated with a spin j puncture is $8\pi l_p^2\beta\sqrt{j(j+1)}$. This, as concluded by Dreyer [15], suggests that only a spin $j = 1$ puncture is lost during the emission of a quantum of energy $\hbar\omega^R_{QNM}$, and also that the Immirzi parameter β takes the value $\beta = \frac{ln3}{2\pi\sqrt{2}}$ instead of the earlier preferred value. This suggests that the underlying group for quantum gravity is SO(3) instead of SU(2). Thus, in the calculation of dimensionality of the boundary Hilbert space formula (1), which also holds for SO(3) conformal field theory with spins restricted to integer values, we need to set $j_i = j_{min} = 1$ on all the punctures. An immediate question to ask is whether that changes the $ln(area)$ correction term to the black hole entropy. We shall demonstrate here that though the Immirzi parameter β has a different value, yet the same $ln(area)$ correction with definite coefficient $-3/2$ obtains even with $j_{min} = 1$ at each puncture.

To see our result, we put $j_i = j_{min} = 1$ for all i in formula (1) and evaluate

$$\mathcal{N} = \frac{2}{k+2}\sum_{r=0}^{k/2}\left[\frac{sin\left(\frac{3(2r+1)\pi}{k+2}\right)}{sin\left(\frac{(2r+1)\pi}{k+2}\right)}\right]^p sin^2\left(\frac{(2r+1)\pi}{k+2}\right) \tag{4}$$

in the limit of large k and large number of punctures p. We may rewrite this expression as

$$\mathcal{N} == \frac{1}{2}\left[3F(p) - F(p+1)\right] \tag{5}$$

where we have used the formulae $sin3x = sinx(1 + 2cos2x)$ and $4sin^2x = 3 - (1 + 2cos2x)$, and defined

$$F(p) = \frac{1}{k+2}\sum_{r=0}^{k/2}\left[1 + 2cos\left(\frac{2(2r+1)\pi}{k+2}\right)\right]^p . \tag{6}$$

This for large k can be approximated by an integral

$$F(p) = \frac{1}{2\pi}\int_0^{2\pi} dx\ (1 + 2cosx)^p . \tag{7}$$

Expanding the integral as powers of $2cosx$ and performing the integration, it is straightforward to see that

$$F(p) = \sum_{l=0}^{[p/2]}\frac{p!}{(p-2l)!(l!)^2} = \sum_{l=0}^{[p/2]}\frac{\Gamma(p+1)}{\Gamma(p-2l+1)\left[\Gamma(l+1)\right]^2} . \tag{8}$$

This sum now needs to be evaluated for large p. This we do by the method of steepest descent. To this effect, we write

$$e^{f(z)} = \frac{\Gamma(p+1)}{\Gamma(p-2z+1)\ \left[\Gamma(z+1)\right]^2} .$$

Thus, expanding $f(z)$ about a maximum, $f(z) = f(z_0) + f''(z_0)\ (z-z_0)^2\ /2 + \cdots$ where the maximum of $f(z)$ occurs at $z = z_0$, the method of steepest descent yields

$$F(p) \simeq e^{f(z_0)}\int dy\ e^{\ f''(z_0)\ y^2\ \ /2} \simeq \sqrt{\frac{2\pi}{-f''(z_0)}}\ \ e^{f(z_0)} .$$

In terms of the Psi function $\Psi(z) = \Gamma'(z)/\Gamma(z)$,

$$\begin{aligned} f'(z) &= 2\,[\ \Psi(p-2z+1)\ -\ \Psi(z+1)\ ] \\ f''(z) &= -2\,[\ 2\Psi'(p-2z+1)\ +\ \Psi'(z+1)\ ]\ . \end{aligned}$$

The maximum is given by $f'(z_0) = 0$ which yields $z_0 = p/3$ and $f''(z_0) = -6\Psi'(z_0)$. Since $\Psi(z) \simeq lnz + \cdots$ for large z, one obtains $f''(z_0) \simeq -\frac{6}{z_0} + \mathcal{O}\left(\frac{1}{z_0^2}\right) \simeq -\frac{18}{p} + \mathcal{O}\left(\frac{1}{p^2}\right)$ in the limit of large p. $F(p)$ is then given by

$$F(p)\ \simeq\ \frac{\Gamma(p+1)}{\left[\Gamma(\frac{p}{3})\right]^3}\ \sqrt{p}\ \left[\ 1\ +\ \mathcal{O}\left(\frac{1}{p}\right)\ \right]\ .$$

For large values of z, the Gamma function $\Gamma(z)$ is given by

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z}\left[1 + \frac{1}{12z} + \cdots\right]$$

This yields $F(p)$ for large values of p:

$$F(p)\ \simeq\ C'\ \frac{3^p}{\sqrt{p}}\ \left[\ 1\ -\ \frac{a}{p}\ +\ \cdots\ \right] \tag{9}$$

where C' and a constants independent of p. Thus, finally, the dimensionality of the boundary Hilbert space of a black hole for large p is

$$\mathcal{N}\ =\ \frac{1}{2}\ [\ 3F(p)\ -\ F(p+1)\ ]\ =\ C\ \frac{3^p}{p^{\frac{3}{2}}}\ \left[\ 1\ +\ \mathcal{O}(\frac{1}{p})\ \right] \tag{10}$$

where C is an irrelevent constant, independent of p. This formula has to be contrasted with that in equation (2) for the case where spin $j_{min} = \frac{1}{2}$ was placed on each puncture. The entropy of the black hole $S = ln\mathcal{N}$ is now

$$S\ =\ p\ ln3\ -\ \frac{3}{2}\ lnp\ +\ \mathcal{O}(p^0)\ . \tag{11}$$

Next, for each spin 1 puncture, the associated area is $\Delta A_H = 8\pi l_p^2 \beta\sqrt{2}$, so that for black hole area A_H, $p = \frac{\beta_0}{\beta}\ \left(\frac{A_H}{4l_p^2}\right)$, $\beta_0 = \frac{1}{2\pi\sqrt{2}}$, and $\beta = \beta_0 ln3 = \frac{ln3}{2\pi\sqrt{2}}$. Now, entropy formula (11) in terms of horison area becomes:

$$S\ =\ \frac{A_H}{4l_p^2}\ -\ \frac{3}{2}\ ln\left(\frac{A_H}{4l_p^2}\right) + \cdots\ , \tag{12}$$

same as the earlier one. Clearly, though the Immirzi parameter has changed by changing the value of minimum spin j_{min} on the punctures from 1/2 to 1, the entropy formula along with its $ln(area)$ correction remains unaltered from those found in reference [8].

We close our discussion with a few remarks. The formula (5) for large k only counts the number of ways SU(2) singlets can be obtained by composing p spin one representations. To see this, we may rewrite the formula as

$$\mathcal{N}\ =\ F(p)\ -\ G(p)$$

where, using ${}^nC_m \equiv \frac{n!}{m!(n-m)!}$, $F(p)$ given in equation (8) may be rewritten as

$$F(p)\ =\ \sum_l\ {}^pC_{2l}\ \ {}^{2l}C_l \tag{13}$$

and

$$G(p)\ =\ \frac{1}{2}\ [\ F(p+1)\ -\ F(p)\ ]\ =\ \sum_l\ \frac{p!}{(p-(2l-1))!\ (l-1)!\ l!}\ =\ \sum_l\ {}^pC_{2l-1}\ \ {}^{2l-1}C_l\ . \tag{14}$$

Now, $F(p)$ and $G(p)$ have a simple interpretation. $F(p)$ counts the number of ways states with net J_3 quantum number $m = 0$ can be obtained by placing $m = 0, \pm 1$ on p punctures. This can be done by picking $2l$ preferred punctures in

${}^pC_{2l}$ ways and then place $m = +1$ on l of them in ${}^{2l}C_l$ ways and on rest l punctures we place $m = -1$. Other $(p-2l)$ punctures carry $m = 0$. Then summing over l from 0 to $[p/2]$ yields the total number of ways ($= F(p)$) net $m = 0$ states can be obtained. But this overcounts the number of SU(2) singlets, because net $m = 0$ states are also contained in possible spin $j = 1, 2, 3, \cdots, p$ representations obtained in the composition of p spin 1 representations. These extra net $m = 0$ states are equal in number to net $m = +1$ (or equivalently $m = -1$) states in spin $j = 1, 2, 3, \cdots, p$ representations of the product. Now states with net $m = +1$ can be obtained by selecting $(2l-1)$ preferred punctures in ${}^pC_{2l-1}$ ways and placing $m = +1$ on l of them in ${}^{2l-1}C_l$ ways and $m = -1$ on the rest $(l-1)$ punctures. Other $(p-(2l-1))$ punctures carry $m = 0$. This is to be done for all values of $l = 0, 1, 2, \cdots$, leading to the fact that $G(p)$ in (14) above counts the number of ways net $m = +1$ (equivalently $m = -1$) states can be obtained. The difference $F(p) - G(p)$ thus does indeed count the number of ways singlets can be obtained by composing p spin 1 representations.

In fact placing any value of spin j_{min} on every puncture, it can be shown, does not change the coefficient of the $ln(area)$ correction for the black hole entropy. Indeed, by a calculation similar to above, the dimensionality of the boundary Hilbert space of such a black hole with j_{min} on every puncture for large p is

$$\mathcal{N} \simeq C \, \frac{(2j_{min} + 1)^p}{p^{\frac{3}{2}}}$$

where C is an irrelevent constant independent of p.

It is now clear that the resultant entropy for any value of j_{min} has the $ln(area)$ correction term, with the definite coefficient $-3/2$. However, Immirzi parameter does depend on the value of j_{min}. In particular, preferred value of $j_{min} = 1$ suggested by equating the mass change corresponding to the minimum change in area $\Delta A_H = 8\pi l_p^2 \beta \sqrt{j_{min}(j_{min}+1)}$ with the energy $\Delta M = \hbar\omega^R_{QNM}$ of a quantum associated with quasinormal mode frequency $\omega^R_{QNM} = \frac{ln3}{8\pi M}$, and corresponding Immirzi parameter $\beta = \frac{ln3}{2\pi\sqrt{2}}$, the entropy of a large area black hole is given by the formula (12), with coefficient $-3/2$ for the $ln(area)$ correction.

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